Rankine–Hugoniot conditions

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The **Rankine–Hugoniot conditions**, also referred to as **Rankine–Hugoniot jump conditions** or **Rankine–Hugoniot relations**, describe the relationship between the states on both sides of a shock wave in a one-dimensional flow. They are named in recognition of the work carried out by Scottish engineer and physicist William John Macquorn Rankine[1] and French engineer Pierre Henri Hugoniot.[2] See also Salas (2006)[3] for some historical background.

Contents

Basics: Euler equations in one dimension

Consider gas in a one-dimensional container (e.g., a long thin tube). Assume that the fluid is inviscid (i.e., it shows no viscosity effects as for example friction with the tube walls). Furthermore, assume that there is no heat transfer by conduction or radiation and that gravitational acceleration can be neglected. Such a system can be described by the following system of conservation laws, known as the 1D Euler equations

$$(1) \qquad \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$$

$$(2) \qquad \frac{\partial \rho u}{\partial t} = -\frac{\partial}{\partial x}\left(\rho u^2 + p\right)$$

$$(3) \qquad \frac{\partial \rho E}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u\left(e + \frac{1}{2}u^2 + p/\rho\right)\right],$$

where

$\rho =$ fluid mass density, [kg/m^3]

$u =$ fluid velocity, [m/s]

$e =$ specific internal energy of the fluid, [J/kg]

$p =$ fluid pressure, [Pa]

$t =$ time, [s]

$x =$ distance, [m], and

$E = e + \frac{1}{2}u^2$, specific total energy of the fluid, [J/kg].

Assume further that the gas is calorically ideal and that therefore a polytropic equation-of-state of the simple form

$$(4) \qquad p = (\gamma - 1)\,\rho e,$$

is valid, where γ is the constant ratio of specific heats c_p/c_v. This quantity also appears as the *polytropic exponent* of the polytropic process described by

$$(5) \qquad \frac{p}{\rho^\gamma} = \text{constant}.$$

For an extensive list of compressible flow equations, etc., refer to NACA Report 1135 (1953).[4]

Note: For a calorically ideal gas γ is a constant and for a thermally ideal gas γ is a function of temperature. In the latter case, the dependence of pressure on mass density and internal energy might differ from that given by equation (4).

The jump condition

Before proceeding further it is necessary to introduce the concept of a *jump condition* – a condition that holds at a discontinuity or abrupt change.

Consider a 1D situation where there is a jump in the scalar conserved physical quantity w, which is governed by the hyperbolic conservation law

$$(6) \qquad \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0.$$

Let the solution exhibit a jump (or shock) at $x = x_s(t)$ and integrate over the partial domain, $x \in (x_1, x_2)$, where $x_1 < x_s(t)$ and $x_s(t) < x_2$,

$$(7) \qquad \frac{d}{dt}\left(\int_{x1}^{x_s(t)} w\,dx + \int_{x_s(t)}^{x2} w\,dx\right) = -\int_{x1}^{x2} \frac{\partial}{\partial x} f(w)\,dx$$

$$(8) \qquad \therefore\ w_1 \frac{dx_s}{dt} - w_2 \frac{dx_s}{dt} + \int_{x1}^{x_s(t)} w_t\,dx + \int_{x_s(t)}^{x2} w_t\,dx = -\left. f(w)\right|_{x1}^{x2}$$

The subscripts *1* and *2* indicate conditions *just upstream* and *just downstream* of the jump respectively. Note, to arrive at equation (8) we have used the fact that $dx_1/dt = 0$ and $dx_2/dt = 0$.

Now, let $x_1 \to x_s(t)$ and $x_2 \to x_s(t)$, when we have $\int_{x1}^{x_s(t)} w_t\,dx \to 0$ and $\int_{x_s(t)}^{x2} w_t\,dx \to 0$, and in the limit

$$(9) \qquad s\,(w_1 - w_2) = f(w_1) - f(w_2),$$

where we have defined $s = dx_s(t)/dt$ (the system *characteristic* or *shock speed*), which by simple division is given by

$$(10) \qquad s = \frac{f(w_1) - f(w_2)}{w_1 - w_2}.$$

Equation (9) represents the jump condition for conservation equation (6). A shock situation arises in a system where its *characteristics* intersect, and under these conditions a requirement for a unique single-valued solution is that the solution should satisfy the *admissibility condition* or *entropy condition*. For physically real applications this means that the solution should satisfy the *Lax entropy condition*

$$f'(w_1) > s > f'(w_2), \tag{11}$$

where $f'(w_1)$ and $f'(w_2)$ represent *characteristic speeds* at upstream and downstream conditions respectively.

Euler equations shock condition

In the case of the hyperbolic conservation equation (6), we have seen that the shock speed can be obtained by simple division. However, for the 1D Euler equations (1), (2) and (3), we have the vector state variable $[\rho, \rho u, \rho E]^T$ and the jump conditions become

$$s(\rho_2 - \rho_1) = \rho_2 u_2 - \rho_1 u_1 \tag{12}$$

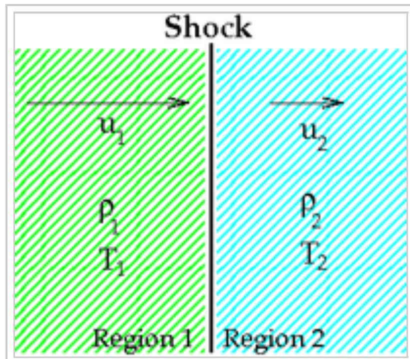
$$s(\rho_2 u_2 - \rho_1 u_1) = \left(\rho_2 u_2^2 + p_2\right) - \left(\rho_1 u_1^2 + p_1\right) \tag{13}$$

$$s(\rho_2 E_2 - \rho_1 E_1) = \left[\rho_2 u_2 \left(e_2 + \frac{1}{2}u_2^2 + p_2/\rho_2\right)\right] - \left[\rho_1 u_1 \left(e_1 + \frac{1}{2}u_1^2 + p_1/\rho_1\right)\right]. \tag{14}$$

Equations (12), (13) and (14) are known as the *Rankine–Hugoniot conditions* for the Euler equations and are derived by enforcing the conservation laws in integral form over a control volume that includes the shock. For this situation s cannot be obtained by simple division. However, it can be shown by transforming the problem to a moving co-ordinate system (setting $s' := s - u_1, u_1' := 0, u_2' := u_2 - u_1$ to remove u_1) and some algebraic manipulation (involving the elimination of u_2' from the transformed equation (13) using the transformed equation (12)), that the shock speed is given by

$$s = u_1 + c_1 \sqrt{1 + \frac{\gamma + 1}{2\gamma}\left(\frac{p_2}{p_1} - 1\right)}, \tag{15}$$

where $c_1 = \sqrt{\gamma p_1/\rho_1}$ is the speed of sound in the fluid at upstream conditions.



A schematic diagram of a shock wave situation with the density ρ, velocity u, and temperature T indicated for each region.

See Laney (1998),[5] LeVeque (2002),[6] Toro (1999),[7] Wesseling (2001),[8] and Whitham (1999)[9] for further discussion.

Stationary shock

For a stationary shock $s = 0$, and for the 1D Euler equations we have

$$\rho_1 u_1 = \rho_2 u_2 \tag{12}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{13}$$

$$\rho_1 u_1 \left(e_1 + \frac{1}{2}u_1^2 + p_1/\rho_1\right) = \rho_2 u_2 \left(e_2 + \frac{1}{2}u_2^2 + p_2/\rho_2\right). \tag{14}$$

In view of equation (12) we can simplify equation (14) to

$$(15) \qquad e_1 + \frac{1}{2}u_1^2 + p_1/\rho_1 = e_2 + \frac{1}{2}u_2^2 + p_2/\rho_2,$$

which is a statement of Bernoulli's principle, under conditions where gravitational effects can be neglected.

Substituting u_1 and u_2 from equations (12) and (13) into equation (15) yields the following relationship:

$$(16) \qquad 2\left(h_2 - h_1\right) = \left(p_2 - p_1\right) \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right),$$

where $h = \frac{p}{\rho} + e$ represents specific enthalpy of the fluid. Eliminating internal energy e in equation (15) by use of the equation-of-state, equation (4), yields

$$(17) \qquad \frac{\rho_2}{\rho_1} = \frac{\frac{p_2}{p_1}(\gamma+1) + (\gamma-1)}{(\gamma+1) + \frac{p_2}{p_1}(\gamma-1)} = \frac{u_1}{u_2}$$

$$(18) \qquad \frac{p_2}{p_1} = \frac{\frac{\rho_2}{\rho_1}(\gamma+1) - (\gamma-1)}{(\gamma+1) - \frac{\rho_2}{\rho_1}(\gamma-1)}.$$

From physical considerations it is clear that both the upstream and downstream pressures must be positive, and this imposes an upper limit on the density ratio in equations (17) and (18) such that $\rho_2/\rho_1 < (\gamma+1)/(\gamma-1)$. As the strength of the shock increases, there is a corresponding increase in downstream gas temperature, but the density ratio ρ_2/ρ_1 approaches a finite limit: 4 for an ideal monatomic gas $(\gamma = 5/3)$ and 6 for an ideal diatomic gas $(\gamma = 1.4)$. Note: air is comprised predominately of diatomic molecules and therefore at standard conditions $\gamma_{\text{air}} \simeq 1.4$.

See also

- Calculate normal shock wave parameters for mixtures of imperfect gases. Gas Dynamics Toolbox (http://web.ics.purdue.edu/~alexeenk/GDT/index.html)
- Shock polar

References

1. ^ Rankine, W. J. M. (1870). "On the thermodynamic theory of waves of finite longitudinal disturbances" (http://gallica.bnf.fr/ark:/12148/bpt6k559658/f328.image.r=Philosophical+Transactions+of+the+Royal+Society+of+London+1870.langEN). *Philosophical Transactions of the Royal Society of London* **160**: 277–288. doi:10.1098/rstl.1870.0015 (http://dx.doi.org/10.1098%2Frstl.1870.0015).
2. ^ Hugoniot, H. (1887). "Mémoire sur la propagation des mouvements dans les corps et spécialement dans les gaz parfaits (première partie) [Memoir on the propagation of movements in bodies, especially perfect gases (first part)]" (http://books.google.com/books?id=wccAAAAAYAAJ&pg=PA3#v=onepage&q&f=false). *Journal de l'École Polytechnique* (in French) **57**: 3–97. See also: Hugoniot, H. (1889) "Mémoire sur la propagation des mouvements dans les corps et spécialement dans les gaz parfaits (deuxième partie)" (http://gallica.bnf.fr/ark:/12148/bpt6k4337130/f5.image) [Memoir on the propagation of movements in bodies, especially perfect gases (second part)], *Journal de l'École Polytechnique*, vol. 58, pages 1-125.
3. ^ Salas, M. D. (2006). "The Curious Events Leading to the Theory of Shock Waves, Invited lecture, *17th Shock Interaction Symposium*, Rome, 4–8 September." (http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/20060047586_2006228914.pdf).
4. ^ Ames Research Staff (1953), "Equations, Tables and Charts for Compressible Flow" (http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19930091059_1993091059.pdf), *Report 1135 of the National Advisory Committee for Aeronautics*
5. ^ Laney, Culbert B. (1998). *Computational Gasdynamics*. Cambridge University Press. ISBN 978-0-521-62558-6.
6. ^ LeVeque, Randall (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press. ISBN 978-0-521-

00924-9.
7. ^ Toro, E. F. (1999). *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer-Verlag. ISBN 978-3-540-65966-2.
8. ^ Wesseling, Pieter (2001). *Principles of Computational Fluid Dynamics*. Springer-Verlag. ISBN 978-3-540-67853-3.
9. ^ Whitham, G. B. (1999). *Linear and Nonlinear Waves*. Wiley. ISBN 978-0-471-94090-6.

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